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Frames of Periodic Shift-Invariant Spaces[1]

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This note studies Bessel sequences and frames of shift-invariant spaces generated by a countable set of periodic functions. We give characterizations under which the set of translations of the countable set is a Bessel sequence or a frame in terms of spectral decompositions of some self-adjoint operators. © 2000 Academic Press

Key Words: Bessel sequence; frame; shift-invariant space.

1. INTRODUCTION

Let $L_2([0,1]^s)$ denote the Hilbert space of all squared integrable functions on $[0,1]^s$ which are $\mathbb{Z}^s$-period. For an $s\times s$ integer and invertible matrix M, we define the operators of translation on $L_2([0,1]^s)$ by

$$T^\alpha f = f(\cdot - M^{-1}\alpha), \qquad \alpha\in\mathbb{Z}^s.$$

A *shift invariant space* $\mathscr{S}$ is a closed subspace of $L_2([0,1]^s)$ if it satisfies

$$f\in\mathscr{S}\Rightarrow T^\alpha f\in\mathscr{S} \qquad \forall f\in\mathscr{S},\ \alpha\in\mathbb{Z}^s.$$

It is known that the shift invariant spaces play an important role in multiresolution analysis and wavelets of $L_2([0,1]^s)$.

Let E be a complete set of representatives of coset of $\mathbb{Z}^s/M\mathbb{Z}^s$, and let Γ be a complete set of representatives of coset of $\mathbb{Z}^s/M^T\mathbb{Z}^s$. Clearly, $\#E=\#\Gamma=m:=|\det M|$. Without loss of generality we assume $0\in E$ and $0\in\Gamma$. Given a set $\Phi=\{\phi_k\}_{k\in\mathbb{N}}\subseteq L_2([0,1]^s)$, we denote by $\mathscr{S}_M(\Phi)$ the closure in $L_2([0,1]^s)$ of the space

$$\left\{f : f=\sum_{\varepsilon\in E}\sum_{k\in\mathbb{N}} c_{\varepsilon,k}T^\varepsilon\phi_k\right\},$$

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where all $c_{\varepsilon, k}$ but finitely many are zero. We are interested in characterizing the Bessel sequence property and the frame property of

$$E_{\Phi} := \{ T^{\varepsilon}\phi_k : \varepsilon \in E, k \in \mathbb{N} \}$$

for $\mathscr{S}_M(\Phi)$.

A sequence $\{x_k\}_{k \in \mathbb{N}}$ in a Hilbert space X is called a *Bessel sequence* for X if there is a constant $\Lambda > 0$ such that, for any $x \in X$,

$$\sum_{k \in \mathbb{N}} |\langle x, x_k \rangle_X|^2 \leqslant \Lambda \, \|x\|_X^2.$$

The *bound* of a Bessel sequence is the smallest number $\Lambda > 0$ that can be used in the above inequality.

We note that any subsequence of a Bessel sequence is also a Bessel sequence for the closed subspace it spans.

A sequence $\{x_k\}_{k \in \mathbb{N}}$ is a *frame* for X if there are constants $\Lambda > 0$ and $\lambda > 0$ satisfying

$$\lambda \, \|x\|_X^2 \leqslant \sum_{k \in \mathbb{N}} |\langle x, x_k \rangle_X|^2 \leqslant \Lambda \, \|x\|_X^2 \qquad \forall x \in X.$$

The *frame bounds* are the largest λ and the smallest Λ that can be used in the above inequalities. A frame with bounds λ and Λ is *tight* if $\lambda = \Lambda$.

For a set Φ of finitely many functions of $L_2(\mathbb{R}^s)$, we gave in [1] a simple characterization for the $\mathbb{Z}^s$-translations $\{\phi(\cdot - \alpha) : \alpha \in \mathbb{Z}^s, \phi \in \Phi\}$ to be a frame for the closed space they span. Independently, Ron and Shen [3] established a more complete characterization for a countable set as well as a finite set. Moreover, the notion of *dual Gramian matrix* was introduced in [3]. A characterization for $\{\phi(\cdot - \alpha) : \alpha \in \mathbb{Z}^s, \phi \in \Phi\}$ being a frame for $L_2(\mathbb{R}^s)$ was given in terms of some dual Gramian matrices ([3]). It is the purpose of this note to find the analog of these results for periodic case. To this end, we make use of spectral decompositions of self-adjoint operators.

2. CHARACTERIZATIONS OF FRAMES IN $L_2([0, 1]^s)$

The bracket product of two functions $f, g \in L_2(\mathbb{R}^s)$ was introduced in [2]. Similarly, for two functions f and g in $L_2([0, 1]^s)$, we define the (periodic) bracket product of f with g by

$$[f, g](\gamma) = \sum_{\beta \in \mathbb{Z}^s} \hat{f}(\gamma + M^T\beta) \, \overline{\hat{g}(\gamma + M^T\beta)}, \qquad \gamma \in \Gamma,$$

where $\hat{f}$ is the Fourier transform of f,

$$\hat{f}(\alpha) = \int_{[0,1]^s} f(x)\, e^{-2\pi\alpha \cdot x}\, dx, \qquad \alpha \in \mathbb{Z}^s.$$

Using the unitarity of matrix

$$\frac{1}{\sqrt{m}} \left(e^{2\pi M^{-1}\varepsilon \cdot \gamma}\right)_{\varepsilon \in E,\, \gamma \in \Gamma}$$

and Parseval's identity, it is easy to prove the following equalities

$$\|f\|^2 = \sum_{\gamma \in \Gamma} [f, f](\gamma) \tag{1}$$

and

$$\sum_{\varepsilon \in E} |\langle f, T^{\varepsilon} g\rangle|^2 = m \sum_{\gamma \in \Gamma} |[f, g](\gamma)|^2. \tag{2}$$

LEMMA 1. *Let E_Φ be a Bessel sequence for $\mathcal{S}_M(\Phi)$ with bound Λ, where $\Phi = \{\phi_k\}_{k \in \mathbb{N}}$. Suppose that $X(\gamma) := \{x_k(\gamma)\}_{k \in \mathbb{N}} \in \ell_2(\mathbb{N})$ for any $\gamma \in \Gamma$. Then for any $\gamma \in \Gamma$ and $\beta \in \mathbb{Z}^s$ the series*

$$\sum_{k \in \mathbb{N}} x_k(\gamma)\, \hat{\phi}_k(\gamma + M^T \beta) \tag{3}$$

converges.

Moreover, let $y(\alpha)$ be the sum of the series in (3), where $\beta \in \mathbb{Z}^s$, $\gamma \in \Gamma$ and $\alpha \in \mathbb{Z}^s$ satisfy $\alpha = \gamma + M^T\beta$. Then there exists a function $f \in \mathcal{S}_M(\Phi)$ satisfying

$$\hat{f}(\alpha) = y(\alpha), \qquad \alpha \in \mathbb{Z}^s, \tag{4}$$

and

$$\|f\|^2 \leqslant \Lambda m^{-1} \sum_{\gamma \in \Gamma} \sum_{k \in \mathbb{N}} |x_k(\gamma)|^2. \tag{5}$$

Proof. For any n, let $\Phi_n := \{\phi_k\}_{k=1}^n$. Being a subsequence of E_Φ, E_{Φ_n} is a Bessel sequence for $\mathcal{S}_N(\Phi_n)$ with bound $\leqslant \Lambda$. For any $n \in \mathbb{N}$ and $\gamma \in \Gamma$, let $G_n(\gamma)$ be the matrix $([\phi_j, \phi_k](\gamma))_{k,j=1}^n$. Clearly, $G_n(\gamma)$ is a self-adjoint matrix. We use $\|G_n(\gamma)\|$ to denote the largest eigenvalue of $G_n(\gamma)$. We claim that

$$\|G_n(\gamma)\| \leqslant \Lambda m^{-1} \qquad \forall \gamma \in \Gamma, \qquad n \in \mathbb{N}. \tag{6}$$

Indeed, the proof of (6) is similar to that of its analogue in nonperiodic case ([1] and [3]). However, for the reader's convenience, we include the argument. We note that a function $f \in \mathcal{S}_M(\Phi_n)$ if and only if there are m vectors $x(\gamma) := (x_1(\gamma), x_2(\gamma), ..., x_n(\gamma))^T \in \mathbb{C}^n$, $\gamma \in \Gamma$, such that

$$\hat{f}(\gamma + M^T\beta) = \sum_{k=1}^{n} x_k(\gamma)\, \hat{\phi}_k(\gamma + M^T\beta), \qquad \gamma \in \Gamma, \quad \beta \in \mathbb{Z}^s.$$

Substituting the above expression into (1) and (2) gives us

$$\|f\|^2 = \sum_{\gamma \in \Gamma} \sum_{j,k=1}^{n} x_j(\gamma)\, G_n(\gamma)\, \overline{x_k(\gamma)} = \sum_{\gamma \in \Gamma} \langle G_n(\gamma)\, x(\gamma), x(\gamma) \rangle$$

and

$$\sum_{k=1}^{n} \sum_{\varepsilon \in E} |\langle f, T^{\varepsilon}\phi_k \rangle|^2 = m \sum_{k=1}^{n} \sum_{\gamma \in \Gamma} \left| \sum_{j=1}^{n} [\phi_j, \phi_k]\, x_j(\gamma) \right|^2$$

$$= m \sum_{\gamma \in \Gamma} \langle (G_n(\gamma))^2\, x(\gamma), x(\gamma) \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the inner product of two vector in $\mathbb{C}^n$. From these equalities we see that E_{Φ_n} is a Bessel sequence with bound Λ is equivalent to

$$x^*(G_n(\gamma))^2\, x \leqslant \Lambda m^{-1} x^* G_n(\gamma)\, x \qquad \forall \gamma \in \Gamma, x \in \mathbb{C}^n.$$

Obviously, it implies (6), as desired.

Define a function $f_n \in L_2([0, 1]^s)$ by its Fourier transform

$$\hat{f}_n(\gamma + M^T\beta) := \sum_{k=1}^{n} x_k(\gamma)\, \hat{\phi}_k(\gamma + M^T\beta).$$

Obviously $f_n \in \mathcal{S}_M(\Phi_n) \subseteq \mathcal{S}_M(\Phi)$. From (6) we have

$$\|f_n\|^2 = \sum_{\gamma \in \Gamma} \sum_{j,k=1}^{n} x_j(\gamma)[\phi_j, \phi_k](\gamma)\, \overline{x_k(\gamma)}$$

$$\leqslant \Lambda m^{-1} \sum_{\gamma \in \Gamma} \sum_{k \in \mathbb{N}} |x_k(\gamma)|^2.$$

Applying the same arguments to $f_N - f_n \in \mathcal{S}_M(\{\phi_k\}_{k=n+1}^{N})$ $N > n$, we have

$$\|f_N - f_n\|^2 \leqslant \Lambda m^{-1} \sum_{\gamma \in \Gamma} \sum_{k=n+1}^{N} |x_k(\gamma)|^2 \to 0$$

as $N, n \to \infty$. On the other hand,

$$\|f_N - f_n\|^2 = \sum_{\beta \in \mathbb{Z}^s} \sum_{\gamma \in \Gamma} \left| \sum_{k=n+1}^{N} x_k(\gamma)\, \hat{\phi}_k(\gamma + M^T\beta) \right|^2.$$

This implies the convergence of (3). Thus we know that $\{f_n\}_{n \in \mathbb{N}}$ converges in $L_2([0,1]^s)$ to some function, say, f. Obviously, $f \in \mathcal{S}_M(\Phi)$ satisfies (4) and (5). The proof is complete.

Remark. For any $\alpha_0 \in \mathbb{Z}^s$, there are $\beta_0 \in \mathbb{Z}^s$, $\gamma_0 \in \Gamma$ such that $\alpha_0 = \gamma_0 + M^T\beta_0$. Choose ε_k for which $\varepsilon_k x_k(\gamma_0)\, \hat{\phi}_k(\gamma_0 + M^T\beta_0) = |x_k(\gamma_0)\, \hat{\phi}_k(\gamma_0 + M^T\beta_0)|$, $k \in \mathbb{N}$. Therefore from (5) we obtain that under the conditions of Lemma 1

$$\sum_{k \in \mathbb{N}} |\hat{\phi}_k(\alpha_0)|^2 \leqslant \Lambda m^{-1}, \qquad \alpha_0 \in \mathbb{Z}^s.$$

Given $\Phi = \{\phi_k\}_{k \in \mathbb{N}}$ and $\gamma \in \Gamma$, we define an operator $G(\gamma)$, at least on the set of finitely supported sequences $x = \{x_k\}$, by

$$\{G(\gamma)\, x\}_k = \sum_{j \in \mathbb{N}} [\phi_j, \phi_k](\gamma)\, x_j, \qquad \gamma \in \Gamma, k \in \mathbb{N}. \tag{7}$$

These operators play an important role in characterizing Bessel sequences and frames.

LEMMA 2. *Suppose that $\Phi = \{\phi_k\}_{k \in \mathbb{N}}$ and E_Φ is a Bessel sequence with bound Λ. Then the operators $G(\gamma)$, $\gamma \in \Gamma$, defined above can be extended to bounded operators on whole space $\ell_2(\mathbb{N})$ with bound Λm^{-1}.*

Proof. It suffices to prove that, for any $\gamma \in \Gamma$, the bound of $G(\gamma)$, as an operator defined on finitely supported sequences, is not larger than Λm^{-1}.

Let $x = \{x_k\}_{k \in \mathbb{N}} \in \ell_2(\mathbb{N})$ satisfy $x_k = 0$ for $k \geqslant n+1$, where n is some integer. For any $N > n$, define a vector $z = (x_1, x_2, ..., x_n, 0, ..., 0)^T \in \mathbb{C}^N$. It is easily seen that for any $k \leqslant N$, $\{G(\gamma)\, x\}_k$ is equal to the kth component of $G_N(\gamma)\, z$. Therefore,

$$\sum_{1 \leqslant k \leqslant N} |\{G(\gamma)\, z\}_k|^2 = \|G_N(\gamma)\, z\|^2 \leqslant (\Lambda m^{-1}\, \|z\|)^2,$$

where the last inequality follows from (6). The proof is complete.

LEMMA 3. *Let $\Phi = \{\phi_k\}_{k \in \mathbb{N}} \subseteq L_2([0,1]^s)$ and E_Φ be a Bessel sequence with bound Λ. Suppose that, for any $\gamma \in \Gamma$, $x(\gamma) = \{x_k(\gamma)\}_{k \in \mathbb{N}}$ is a finitely* *supported sequence. Then, for* $f \in \mathcal{S}_M(\Phi)$ *defined in* (4) *by* $x(\gamma)$, $\gamma \in \Gamma$, *we have*

$$\|f\|^2 = \sum_{\gamma \in \Gamma} \langle G(\gamma))\, x(\gamma), x(\gamma) \rangle \tag{8}$$

and

$$\sum_{\varepsilon \in E} \sum_{k \in \mathbb{N}} |\langle f, T^{\varepsilon} \phi_k \rangle|^2 = m \sum_{\gamma \in \Gamma} \langle (G(\gamma))^2\, x(\gamma), x(\gamma) \rangle. \tag{9}$$

Proof. Recall that $\hat{f}(\gamma + M^T \beta) = \sum_{k \in \mathbb{N}} x_k(\gamma)\, \hat{\phi}_k(\gamma + M^T \beta)$. Substituting it into (1) and changing the order of summations we get as in the proof of (6)

$$\|f\|^2 = \sum_{\gamma \in \Gamma} \langle G(\gamma)\, x(\gamma), x(\gamma) \rangle.$$

Note that the order of summations may be changed due to $x(\gamma)$, $\gamma \in \Gamma$, being finitely supported.

Similarly, by substituting the expression of $\hat{f}$ into (2) we have

$$\sum_{\gamma \in \Gamma} \sum_{j \in \mathbb{N}} |\langle f, T^l \phi_j \rangle|^2 = m \sum_{\gamma \in \Gamma} \langle G(\gamma)\, x(\gamma), G(\gamma)\, x(\gamma) \rangle.$$

Therefore (9) is true by the fact that $G(\gamma)$, $\gamma \in \Gamma$, are self-adjoint operators. The proof is complete.

Now we are in the position to prove our main result of this section.

THEOREM 1. *Suppose that* $\Phi = \{\phi_k\}_{k \in \mathbb{N}} \subseteq L_2([0, 1]^s)$.

(i) *If* E_Φ *is a frame with frame bounds* λ *and* Λ, *then the operators* $G(\gamma)$ *defined by* (7) *can be extended to whole* $\ell_2(\mathbb{N})$ *and satisfy that for* $\gamma \in \Gamma$ *and* $x \in \ell_2(\mathbb{N})$

$$\lambda m^{-1} \langle G(\gamma)\, x, x \rangle \leqslant \langle (G(\gamma))^2\, x, x \rangle \leqslant \Lambda m^{-1} \langle G(\gamma)\, x, x \rangle. \tag{10}$$

(ii) *Conversely, if* $G(\gamma)$, $\gamma \in \Gamma$, *are meaningful and satisfy* (10) *for some positive numbers* λ *and* Λ, *then* E_Φ *is a frame for* $\mathcal{S}_M(\Phi)$. *The bounds of the frame* λ *and* Λ *are the best possible numbers satisfying* (10).

Proof. (i) follows immediately from Lemma 3.

For the proof of (ii) we note that all the functions $f \in \mathcal{S}_M(\Phi)$ defined in Lemma 1 for finitely supported $x(\gamma) \in \ell_2(\mathbb{N})$, $\gamma \in \Gamma$, are dense in $\mathcal{S}_M(\Phi)$. Therefore (8), (9) and (10) imply (ii). The proof is complete.

Since all operators $G(\gamma)$, $\gamma \in \Gamma$, are positive on $\ell_2(\mathbb{N})$, we have spectral families $E_t(\gamma)$ such that for $n = 1, 2, \ldots,$

$$\langle (G(\gamma))^n x, x \rangle = \int_{\sigma(G(\gamma))} t^n \, d \, \|E_t(\gamma) x\|^2, \qquad x \in \ell_2(\mathbb{N}), \quad \gamma \in \Gamma, \tag{11}$$

where $\sigma(G(\gamma)) \subseteq [0, \infty)$ is the spectrum set of $G(\gamma)$.

Appealing to (11) we can easily see that the inequalities in (10) are equivalent to

$$\lambda m^{-1} \leqslant \lambda^+(G(\gamma)) \leqslant \|G(\gamma)\| \leqslant \Lambda m^{-1}, \tag{12}$$

where $\lambda^+(A) = \inf\{\lambda : \lambda > 0, \lambda \in \sigma(A)\}$ and $\|A\|$ denotes the norm of operator A.

Now we may restate Theorem 1 in terms of spectra.

THEOREM 1'. *E_Φ is a frame with frame bounds λ and Λ if and only if for each $\gamma \in \Gamma$ the operator $G(\gamma)$ satisfies* (12) *and λ and Λ are the best possible numbers such that* (12) *holds.*

3. FUNDAMENTAL FRAMES FOR $L_2([0, 1]^s)$

In this section we consider briefly the problem when E_Φ is a *fundamental frame*, i.e., a frame for $L_2([0, 1]^s)$, where $\Phi = \{\phi_k\}_{k \in \mathbb{N}} \subseteq L_2([0, 1]^s)$.

For any $f \in L_2([0, 1]^s)$, we have by (1) and (2) that

$$\|f\|^2 = \sum_{\gamma \in \Gamma} \langle \hat{f}|_\gamma, \hat{f}|_\gamma \rangle, \tag{13}$$

and

$$\sum_{\gamma \in \Gamma} |\langle f, T^\varepsilon \phi_k \rangle^2 = m \sum_{\gamma \in \Gamma} \langle \tilde{G}_{\phi_k}(\gamma) \hat{f}|_\gamma, \hat{f}|_\gamma \rangle, \tag{14}$$

where $\hat{f}|_\gamma = \{\hat{f}(\gamma + M^T\beta)\}_{\beta \in \mathbb{Z}^s} \in l_2(\mathbb{Z}^s)$ and $\tilde{G}_{\phi_k}(\gamma)$ is an operator defined, at least on finitely supported sequences, by the matrix $(\hat{\phi}_k(\gamma + M^T\beta) \times \overline{\hat{\phi}_k(\gamma + M^T\beta')})_{\beta, \beta' \in \mathbb{Z}^s}$. It is easily seen that $\tilde{G}_{\phi_k}(\gamma)$ is a positive operator on $\ell_2(\mathbb{Z}^s)$. Therefore E_Φ is a fundamental frame with frame bounds λ and Λ if and only if

$$\lambda m^{-1} \sum_{\gamma \in \Gamma} \langle \hat{f}|_\gamma, \hat{f}|_\gamma \rangle \leqslant \sum_{\gamma \in \Gamma} \sum_{k \in \mathbb{N}} \langle \tilde{G}_{\phi_k}(\gamma) \hat{f}|_\gamma, \hat{f}|_\gamma \rangle \leqslant \Lambda m^{-1} \sum_{\gamma \in \Gamma} \langle \hat{f}|_\gamma, \hat{f}|_\gamma \rangle.$$

If E_Φ is Bessel sequence with bound Λ then, for any $\alpha \in \mathbb{Z}^s$, $\sum_{k \in \mathbb{N}} |\hat{\phi}_k(\alpha)|^2 \leqslant \Lambda m^{-1}$ by Remark. Consequently, every series

$$\sum_{k \in \mathbb{N}} \hat{\phi}_k(\gamma + M^T\beta) \overline{\hat{\phi}_k(\gamma + M^T\beta')}, \qquad \beta, \beta' \in \mathbb{Z}^s,$$

converges absolutely. In this case we may define positive operators $\tilde{G}(\gamma)$, $\gamma \in \Gamma$, at least on finitely supported sequences, by letting

$$\tilde{G}(\gamma) := \left(\sum_{k \in \mathbb{N}} \hat{\phi}_k(\gamma + M^T\beta) \overline{\hat{\phi}_k(\gamma + M^T\beta')} \right)_{\beta, \beta' \in \mathbb{Z}^s}.$$

Recall that $\tilde{G}(\gamma)$ is the analogue of dual Gramian matrix defined in [3].

By the same arguments as before and appealing to equalities (13) and (14) we have

THEOREM 2. *E_Φ is a fundamental frame with frame bounds λ and Λ if and only if, for each $\gamma \in \Gamma$, $\tilde{G}(\gamma)$ is well-defined and satisfies*

$$\lambda m^{-1} \langle x, x \rangle \leqslant \langle \tilde{G}(\gamma)\, x, x \rangle \leqslant \Lambda m^{-1} \langle x, x \rangle \qquad \forall x \in \ell_2(\mathbb{N}). \tag{15}$$

Consequently, E_Φ is both a fundamental frame and a tight frame if and only if

$$\sum_{k \in \mathbb{N}} \hat{\phi}_k(\gamma + M^T\beta) \overline{\hat{\phi}_k(\gamma + M^T\beta')} = \text{cont.}\, \delta_{\beta, \beta'}, \qquad \beta, \beta' \in \mathbb{Z}^s, \quad \gamma \in \Gamma.$$

We note that the inequalities in (15) are equivalent to

$$\lambda m^{-1} \leqslant \|\tilde{G}^{-1}(l)\|^{-1} \leqslant \|\tilde{G}(\gamma)\| \leqslant \Lambda m^{-1}. \tag{16}$$

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